Real-time Frequency Estimation of Local Oscillators Using GPS Timing Receivers

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***Abstract*— In this paper, we show that recently designed the finite impulse response (FIR) filtering algorithm provides real-time estimation of the fractional frequency offset of a local clock with high accuracy via GPS-based measurements of the time interval error (TIE). Such a splendid property of the algorithm is due to small produced noise in the estimates of the TIE. It is also shown that the algorithm applied to sawtooth measurements serves as a nice estimator of the Allan deviation. The latter cannot be estimated via sawtooth-less measurements.**

I. Introduction

Efficient measurements of time interval errors (TIEs) of local clocks are provided using commercially available Global Positioning System (GPS) timing receivers. A typical measurement set utilizing such a receiver is shown in Fig. 1. Here, the TIE $x(n)$, where n are discrete points following with the time step $\tau = 1\text{s}$, between the GPS time and the local clock time is measured in the presence of typically the sawtooth noise $v(n)$ induced by the receiver owing to the principle of the one pulse per second (1PPS) signal formation utilized to the receiver. The standard deviation of $v(n)$ using commercially available receivers is about 30 ns, can reach 10-20 ns [1] and may be improved by removal of systematic errors to no less than 3-5 ns [1], [2].

In modern receivers, such as the Motorola M12+ [3] and SynPaQ III GPS Sensor, uniformly distributed the sawtooth noise $v(n)$ ranges within the bounds $\pm\Delta$ as shown in Fig. 1b. It can be shown that the sawtooth structure of $v(n)$ represents the modulo 2Δ Brownian TIE associated with the phase of a Local Time Clock (LTC) of the receiver. The bound Δ is calculated as a reciprocal of the frequency f_{LTC} of the LTC by $\Delta[\text{ns}] = 10^3/2f_{\text{LTC}}[\text{MHz}]$. In the SynPaQ III GPS Sensor and Motorola M12+ GPS timing receiver, the frequency is chosen to be $f_{\text{LTC}} = 10$ MHz and the bound is thus $\Delta = 50$ ns. In a modified M12+, they let $f_{\text{LTC}} = 40$ MHz [4] and hence $\Delta = 12.5$ ns.

Because noise in the 1PPS signal substantially exceeds short-time fluctuations featured to precision oscillators (these fluctuations are in picoseconds), estimators are applied to GPS-based measurements. A real-time estimator allows evaluating all of the local clock states such as the TIE (first state), fractional frequency offset (second state), linear fractional frequency drift rate (third state), etc. However, two first states are typically of prime importance. For time synchronization of local clocks, a 1PPS signal from the GPS receiver is generally used as an input to a time interval counter to measure the first state (TIE) with the above mentioned time uncertainty. For frequency measurements (second state), a frequency output from a GPS disciplined oscillator (GPSDO) is used as an input to a phase comparator or external time base for a frequency counter [5]. The typical uncertainty of GPS measurements of frequency for 24 hours is $< 2 \times 10^{-13}$.

In this paper we show that GPS-based measurements of the TIE can efficiently be used to provide real-time unbiased finite impulse response (FIR) estimation of the fractional frequency offset avoiding direct measurements using the GPSDO.

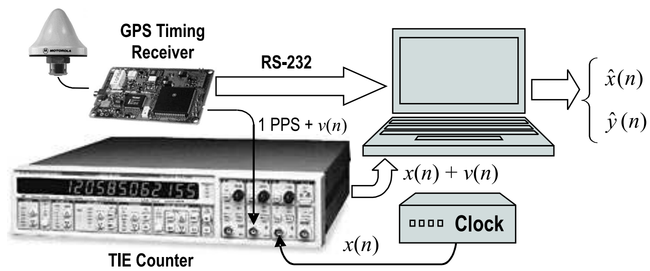

Fig. 1. GPS-based estimation of the fractional frequency offset $\hat{y}_n$.

II. Real Time Unbiased FIR Estimation of Clock States

To provide robust estimation of the TIE linear model via the GPS-based measurements, an unbiased FIR filter was designed and studied in [6]. Soon after, the filter was examined in [7] for applications in GALILEO and it was experimentally shown (Fig. 8 in [7]) that the mean square error (MSE) and Allan deviation in its estimate are both minimum among other estimates examined. Because the TIE model is not always linear, the unbiased FIR approach was recently elaborated in [8] for an arbitrary degree TIE model and a general algorithm was proposed. Some results of the algorithm optimization are given in [9]. Below, we present in brief the approach related to GPS-based measurements of the local clock TIE.

A. TIE Model of a Local Clock with Single Measurements

If we consider a clock on a horizon of N discrete time points starting from zero and think that the clock TIE model is represented with $K + 1$ states, then the clock state and observation equations can be written as follows, respectively,

$$\lambda(n) = \mathbf{B}(n)\lambda(0)\,, \tag{1}$$

$$\xi(n) = \mathbf{C}\lambda(n) + v(n)\,, \tag{2}$$

where $\lambda(n) = [x_1(n)x_2(n)\ldots x_{K+1}(n)]^T$ is a $(K+1)\times 1$ vector of the clock states, $\xi(n)$ is a single GPS-based measurement of the TIE, a time-varying $(K+1)\times(K+1)$ system matrix is

$$\mathbf{B}(n) = \begin{bmatrix} 1 & n\tau & n^2\tau^2/2 & \ldots & (n\tau)^K/K! \\ 0 & 1 & n\tau & \ldots & (n\tau)^{K-1}/(K-1)! \\ 0 & 0 & 1 & \ldots & (n\tau)^{K-2}/(K-2)! \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}, \tag{3}$$

the $1\times(K+1)$ measurement matrix is

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}, \tag{4}$$

and $v(n)$ is noise induced by the receiver, temporary GPS time uncertainty [10], and signal propagation [11].

B. Unbiased FIR Estimates of Clock States

To provide real-time estimation of $\lambda(n)$ and obtain the $(K+1)\times 1$ vector $\hat{\lambda}(n) = [\hat{x}_1(n)\hat{x}_2(n)\ldots\hat{x}_{K+1}(n)]^T$ of estimates $\hat{x}_k(n)$, $k \in [1, K+1]$, of clock states $x_k(n)$, the unbiased FIR filtering algorithm [8] can be used. By this algorithm, the observation vector $\xi(n) = [\xi_1(n)\xi_2(n)\ldots\xi_{K+1}(n)]^T$ is formed, in which the p-component is provided as the backward discrete time derivative of the $(k-1)$-order estimate $\hat{x}_{k-1}(n)$,

$$\xi_k(n) = \frac{\hat{x}_{k-1}(n) - \hat{x}_{k-1}(n-1)}{\tau}\,. \tag{5}$$

Utilizing N points of the nearest past, an unbiased FIR estimate $\hat{x}_k(n)$ of $x_k(n)$ is obtained by the discrete-time convolution applied to (5),

$$\begin{aligned} \hat{x}_k(n) &= \mathcal{C}_k\xi_k(n) \\ &= \sum_{i=0}^{N-1} h_{K+1-k}(i,N)\xi_k(n-i)\,, \end{aligned} \tag{6}$$

where $\mathcal{C}_k$ is the discrete convolution operator and a generic FIR component $h_l(i,N)$, $l \in [0,K]$, has inherent properties

$$h_l(i,N) = \begin{cases} h_l(i,N), & 0 \le i \le N-1 \\ 0, & \text{otherwise} \end{cases}, \tag{7}$$

$$\sum_{i=0}^{N-1} h_l(i,N) = 1\,. \tag{8}$$

The $(K+1)\times 1$ vector $\hat{\lambda}(n)$ of the estimates of the clock states is thus defined as

$$\hat{\lambda}(n) = \mathcal{C}\xi(n)\,, \tag{9}$$

where the $(K+1)\times(K+1)$ convolution operator matrix is

$$\mathcal{C} = \begin{bmatrix} \mathcal{C}_1 & 0 & \ldots & 0 \\ 0 & \mathcal{C}_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathcal{C}_{K+1} \end{bmatrix}, \tag{10}$$

having the components specified by (6).

C. Unique Unbiased FIRs for Real Time Estimation

The unique FIR for real time unbiased estimation was derived in [8] in the form of

$$h_l(i) = \sum_{j=0}^{l} a_{jl} i^j, \tag{11}$$

where a generic coefficient a_{jl} is determined by

$$a_{jl} = (-1)^j \frac{M_{(j+1)1}}{|\mathbf{D}|} \tag{12}$$

via the determinant $|\mathbf{D}|$ and minor $M_{(j+1)1}$ of the $(l+1)\times(l+1)$ quadratic matrix $\mathbf{D}$,

$$\mathbf{D} = \begin{bmatrix} d_0 & d_1 & d_2 & \ldots & d_l \\ d_1 & d_2 & d_3 & \ldots & d_{l+1} \\ d_2 & d_3 & d_4 & \ldots & d_{l+2} \\ \vdots & \vdots & \vdots & \ddots & \\ d_l & d_{l+1} & d_{l+2} & \ldots & d_{2l} \end{bmatrix}, \tag{13}$$

which component d_m, $m \in [0, 2l]$, is calculated by the Bernoulli polynomials $B_n(x)$ as

$$d_m = \sum_{i=0}^{N_l-1} i^m = \frac{1}{m+1}\left[B_{m+1}(N) - B_{m+1}\right], \tag{14}$$

where $B_n = B_n(0)$ is the Bernoulli coefficient.

In time scales, the degree K is identified for the filter memory on a horizon $[0, N-1]$ by the clock precision. Typically, $K=0$ fits cesium clocks, $K=1$ is appropriate for rubidium clocks and $K \in [1,2]$ for crystal clocks. However, $K=3$ may be required for low precision crystal clocks. Covering all these cases, the unique FIRs were found in [8] as

$$h_0(i,N) = \frac{1}{N}\,, \tag{15}$$

$$h_1(i,N) = \frac{2(2N-1) - 6i}{N(N+1)}\,, \tag{16}$$

$$h_2(i,N) = \frac{3(3N^2 - 3N + 2) - 18(2N-1)i + 30i^2}{N(N+1)(N+2)}\,, \tag{17}$$

$$\begin{aligned} h_3(i,N) &= \frac{8(2N^3 - 3N^2 + 7N - 3)}{N(N+1)(N+2)(N+3)} \\ &- \frac{20(6N^2 - 6N + 5)i - 120(2N-1)i^2 + 140i^3}{N(N+1)(N+2)(N+3)}\,. \end{aligned} \tag{18}$$

The uniform FIR (15) corresponds to simple averaging that is optimal in the sense of the minimum produced noise. Its practical usefulness was proved in GPS-based common view measurements [12].

In order to provide estimation of the Allan deviation and fractional frequency offset of a local clock via GPS-based measurements of the TIE, below we will be interested of only two clock states, $x(n) \equiv x_1(t)$ and $y(n) \equiv x_2(n)$.

III. Allan Variance Estimation

Provided the unbiased FIR estimate of the TIE $x(n)$, by (6),

$$\begin{aligned} \hat{x}(n) &= \sum_{i=0}^{N-1} h_K(i,N)\xi(n-i) \\ &= \mathbf{W}_K^T \Xi_x(n)\,, \end{aligned} \tag{19}$$

where the K-degree FIR matrix is

$$\mathbf{W}_K = \begin{bmatrix} h_K(0) \\ h_K(1) \\ \vdots \\ h_K(N_K-1) \end{bmatrix} \tag{20}$$

and a matrix of TIE measurements is

$$\Xi_x(n) = \begin{bmatrix} \xi(n) \\ \xi(n-1) \\ \vdots \\ \xi(n-N_K+1) \end{bmatrix}, \tag{21}$$

the Allan deviation of a local clock oscillator can straightforwardly be estimated with, as suggested in [13] and [14],

$$\sigma_y(\tau) = \sqrt{\frac{1}{2(M-2)\tau^2}\sum_{k=1}^{M-2}[\hat{x}(k+2) - 2\hat{x}(k+1) + \hat{x}(k)]^2} \tag{22}$$

where M is a number of points in the average, and the estimate of the TIE.

In what follows, we use $h_1(i,N)$ given by (16) for a linear TIE model and $h_2(i,N)$, (17), for a quadratic model. To realize, which measurements (sawtooth or sawtooth-less) provide estimation of the Allan deviation with lesser ambiguity, we use the GPS timing SynPaQ III Sensor and Stanford Frequency Counter SR620 to measure the TIE of a local clock imbedded to SR620. To obtain a reference trend, the TIE is simultaneously measured for the Symmetricom cesium standard of frequency CsIII.

A. Allan Variance Estimation via Sawtooth Measurements

In the first experiment, we estimate the Allan deviation of the unbiased FIR estimates provided via the sawtooth measurements. Fig. 2 illustrates the result obtained with the linear FIR (16), $K = 1$, and quadratic FIR (17), $K = 2$.

The most important conclusion is that the estimate of the Allan deviation obtained with a linear FIR (16) and $N \cong$ 7000 behaves very closely to the actual measurements over all values of τ (Fig. 2a). A similar picture (Fig. 2b) is provided by a quadratic FIR (17). In the latter case, however, a horizon N appears to be much larger, of about 1.6×10^4, that substantially increases computation time.

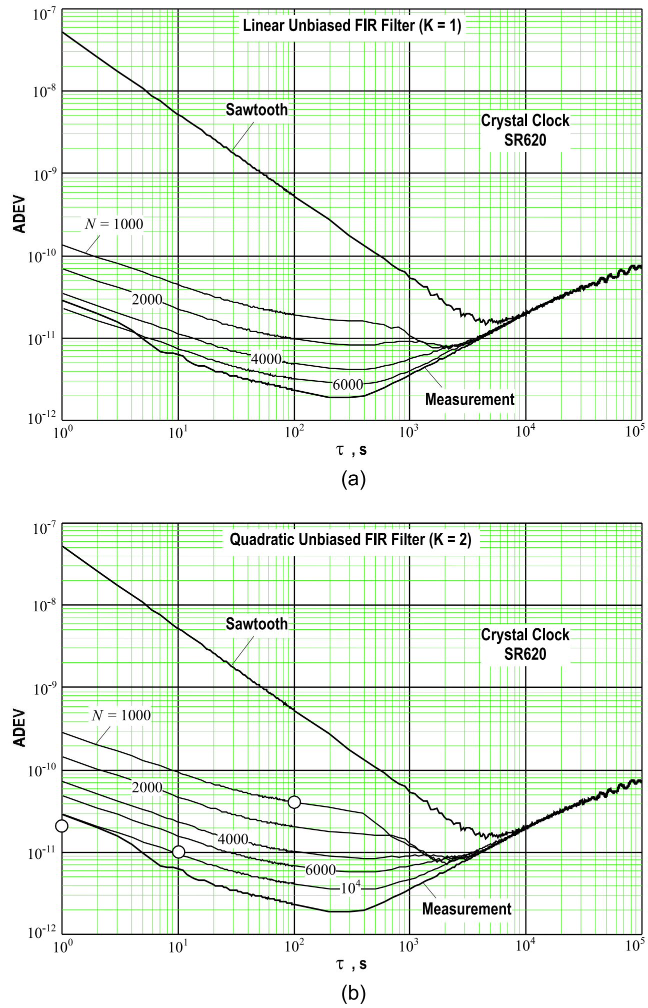

Fig. 2. Estimates of the Allan deviation via GPS-based sawtooth measurements with the unbiased FIR filter: (a) $K = 1$ and (b) $K = 2$.

B. Allan Variance Estimation via Sawtooth-less Measurements

A picture changes cardinally when we apply an estimator to sawtooth-less measurements. Sawtooth noise is efficiently suppressed here by correction. Therefore, the Allan deviation behaves lower than for measurements with sawtooth (Fig. 3). It is seen that neither a linear FIR (16) (Fig. 3a) nor a quadratic FIR (17) (Fig. 3b) is able to save the noise structure and the estimates of the Allan deviation do not fit actual measurements.

Overall, observing Figs. 2 and 3, we infer that GPS-based sawtooth measurements can efficiently be used in estimating the Allan deviation of a local clock and that the sawtooth-less measurements cannot serve with this aim. The major problem

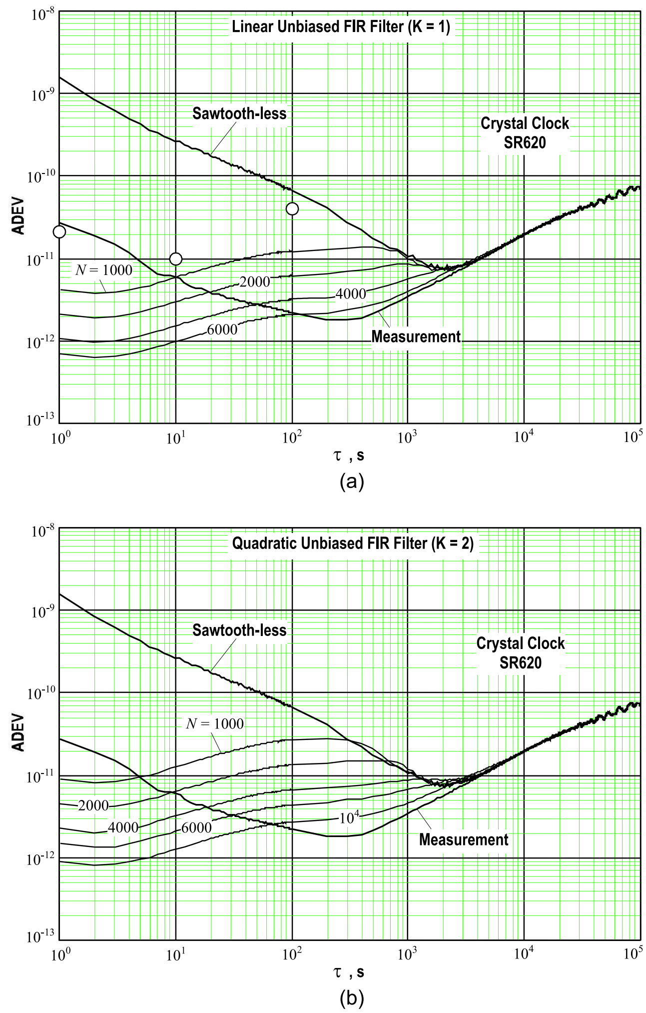

Fig. 3. Estimates of the Allan deviation via GPS-based sawtooth-less measurements with the unbiased FIR filter: (a) $K = 1$ and (b) $K = 2$.

here is, as follows from Fig. 2, to find a proper horizon N in order to minimize errors in estimates of the Allan deviation.

IV. Estimation of Fractional Frequency Offset

By the algorithm described in [8], the fractional frequency offset is estimated as follows. Identified the clock degree K, we first estimate the TIE on a horizon of N_K points by

$$\hat{x}(n) = \mathbf{W}_K^T \mathbf{\Xi}_x(n)\,. \tag{23}$$

The measurement for the second clock state $y(n)$ is then obtained by the backward discrete time derivative of $\hat{x}_n$ as

$$\xi_y(n) = \frac{\hat{x}(n) - \hat{x}(n-1)}{\tau} \tag{24}$$

and the estimate $\hat{y}(n)$ is provided in a like manner on a horizon of N_{K-1} points using $h_{K-1}(i, N_{K-1})$; that is in a batch form,

$$\hat{y}(n) = \sum_{j=0}^{N_{K-1}-1} h_{K-1}(j, N_{K-1})\xi_y(n-j)\,. \tag{25}$$

Finally, employing measurements of the TIE, the fractional frequency offset is estimated in the double batch form by

$$\hat{y}(n) = \frac{1}{\tau}\sum_{j=0}^{N_{K-1}-1}\sum_{i=0}^{N_K-1} h_{K-1}(j, N_{K-1})h_K(i, N_K)$$

$$\times\left[\xi(n-j-i) - \xi(n-j-i-1)\right]\,. \tag{26}$$

In a short matrix form, (24) can be represented as

$$\xi_y(n) = \frac{1}{\tau}\mathbf{W}_K^T\left[\mathbf{\Xi}_x(n) - \mathbf{\Xi}_x(n-1)\right] \tag{27}$$

and the estimate (25) by

$$\hat{y}(n) = \mathbf{W}_{K-1}^T\mathbf{\Xi}_y(n)\,, \tag{28}$$

where

$$\mathbf{\Xi}_y(n) = \frac{1}{\tau}\begin{bmatrix} \mathbf{W}_K^T\left[\mathbf{\Xi}_x(n) - \mathbf{\Xi}_x(n-1)\right] \\ \mathbf{W}_K^T\left[\mathbf{\Xi}_x(n-1) - \mathbf{\Xi}_x(n-2)\right] \\ \vdots \\ \mathbf{W}_K^T\left[\mathbf{\Xi}_x(n-N_K+1) - \mathbf{\Xi}_x(n-N_K)\right] \end{bmatrix} \tag{29}$$

Regarding the feasible TIE models, three particular realizations of the algorithm (26), and so (28), are available.

A. Linear TIE Model

A linear TIE model ($K = 1$) fits well rubidium clocks and is often applied to crystal clocks with oven controlled crystal oscillators (OCXOs). By $K = 1$, (26) becomes

$$\hat{y}(n) = \frac{1}{N_0\tau}\sum_{j=0}^{N_0-1}\sum_{i=0}^{N_1-1} h_1(i, N_1)$$

$$\times\left[\xi(n-j-i) - \xi(n-j-i-1)\right]\,, \tag{30}$$

where $h_1(i, N_1)$ is specified by (16). It can be shown that the first correct value of $\hat{y}(n)$ appears at $n = N_0 + N_1 - 1$.

B. Quadratic TIE Model

Some OCXO-based clocks have a quadratic TIE model ($K = 2$). In this case, (24) is written as

$$\hat{y}(n) = \frac{1}{\tau}\sum_{j=0}^{N_1-1}\sum_{i=0}^{N_2-1} h_1(j, N_1)h_2(i, N_2)$$

$$\times\left[\xi(n-j-i) - \xi(n-j-i-1)\right]\,, \tag{31}$$

having the first correct estimate at $n = N_0 + N_1 + N_2 - 1$. The FIRs for (31) are specified by (16) and (17).

C. Cubic TIE Model

In a like manner, we arrive at the estimate of $y(n)$ for the cubic model, $K = 3$,

$$\hat{y}(n) = \frac{1}{\tau} \sum_{j=0}^{N_2-1} \sum_{i=0}^{N_3-1} h_2(j, N_2) h_3(i, N_3) \times [\xi(n-j-i) - \xi(n-j-i-1)] \ , \quad (32)$$

using the FIRs provided by (17) and (18).

As it follows, to compute the estimate of $y(n)$, one needs specifying two horizons, N_K and N_{K-1}, that may be done at the early stage in the sense of the minimum MSE if to test the clock by the reference oscillator.

D. GPS-based Estimations of Fractional Frequency Offset of a Crystal Clock

To estimate the second state $y(n)$ of a crystal clock imbedded to SR620, the TIE model first needs to be identified in the sense of the minimum MSE for reference measurements. An observation of Fig. 2 shows that either linear model ($K = 1$) or quadratic model ($K = 2$) fits the clock. Letting $K = 2$, estimation of $y(n)$ is provided by the algorithm (31).

For the comparison, we also estimate this state with the three-state Kalman filter in the form given in [8]. To use the filter properly, uniformly distributed the sawtooth noise is approximated by the Gaussian low with the variance $\Delta^2/3$. The q's components of the clock noise correlation matrix are determined via the Allan variance as suggested in [15] to fit actual measurements of the Allan deviation obtained for our sample of SR620: 2.0×10^{-11} at $\tau = 1$s, 8.0×10^{-12} at $\tau = 10$s, 3.0×10^{-12} at $\tau = 100$s, 5.0×10^{-12} at $\tau = 10^3$s, and 2.0×10^{-11} at $\tau = 10^4$s.

Fig. 4 shows the results of measurements and estimations for $N_1 = 4000$. As can be seen (Fig. 4a), both the unbiased FIR estimate $\hat{y}$ and Kalman estimate $\hat{y}_K$ fit well reference measurements and the time derivative of the estimates obtained via the GPS-based sawtooth measurements of the TIE. The goodness-of-fit estimates is demonstrated in Fig. 4b by the Allan deviation. It is seen that the unbiased FIR estimate has much lower noise than in the Kalman estimate in the region up to about $\tau = 10^3$s. Herewith, the Kalman filter demonstrates a bit better performance when $10^3 < \tau < 4 \times 10^3$s.

We notice that the unbiased FIR filter applied to the sawtooth-less measurements also produces the result that fits measurements. However, the noise structure is violated both in the unbiased FIR and Kalman estimates. Accordingly, the Allan deviation in either case behaves similarly to the trends shown in Fig. 3.

V. Conclusions

In this paper, we investigated the trade-off between the unbiased FIR estimates obtained via the GPS-based sawtooth and sawtooth-less measurements. The batch and matrix form algorithms are presented. We show experimentally that the noise structure of the local clock is almost fully saved in the unbiased FIR estimates provided via the sawtooth measurements. Contrary, the sawtooth-less measurements cannot be used to estimate the Alan deviation in the estimates of the TIE. We also demonstrate that the Kalman filter produces more noise in the estimate. The latter can be explained by the sawtooth noise that is not Gaussian.

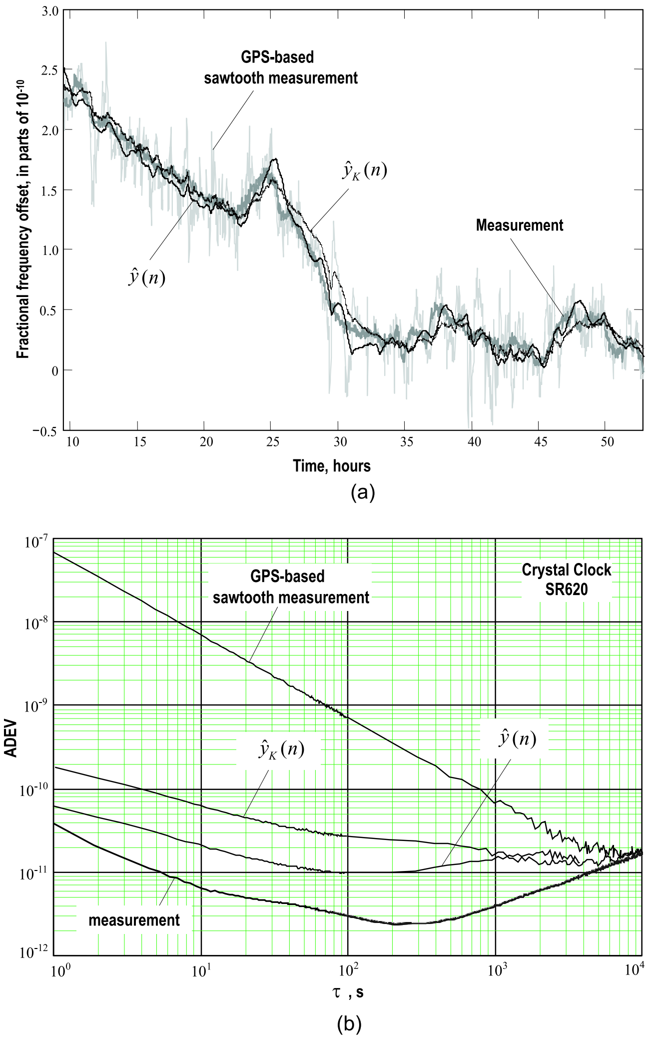

Fig. 4. GPS-based estimation of the fractional frequency offset $\hat{y}_n$: (a) measurements and estimations and (b) Allan deviation.

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